Sec 1.5 1^{st} order linear eqs (part 2)

Last time: $y' + Py = Q$

- complementing/integrating factor $\rho = e^{\int P\,dx}$

$(\rho' = \rho P)$

complement $\rho y' + \rho' y = \rho Q$

$(\rho y)' = \rho Q$

integrate & solve to get

$$\boxed{y = \frac{\int \rho Q\,dx + C}{\rho}}$$

Ex Find a general solution for

$$(x^2+1)y' + 3xy = 6x$$

Eq is 1^{st} order linear, but not in standard form: change to

$$y' + \frac{3x}{x^2+1}y = \frac{6x}{x^2+1}$$

$$P = \frac{3x}{x^2+1}, \quad Q = \frac{6x}{x^2+1}$$

$$\rho = e^{\int P\,dx} = e^{\int \frac{3x}{x^2+1}dx} = e^{\frac{3}{2}\ln|x^2+1| + C}$$

• can pick any simple ρ of that form,
so pick $C=0$ * (see note below)

• Because we can pick simple ρ, we can leave out abs vals for ln.

$$e^{\frac{3}{2}\ln|x^2+1| + \cancel{C}^{\,0}} = e^{\frac{3}{2}\ln(x^2+1)} = \boxed{(x^2+1)^{3/2}} = \rho$$

$$\underline{P} = \frac{3x}{x^2+1}, \quad Q = \frac{6x}{x^2+1}$$

$$\int (\rho y)' \, dx = \int \rho Q \, dx$$

$$\rho y = \int \rho Q \, dx + C$$

$$(x^2+1)^{3/2} y = \int (x^2+1)^{3/2} \frac{6x}{x^2+1} \, dx + C$$

$$= \int 6x(x^2+1)^{1/2} dx + C$$

$$= \cdots$$

$$(x^2+1)^{\frac{3}{2}} y = 2(x^2+1)^{3/2} + C$$

$$\boxed{y = 2 + C(x^2+1)^{-3/2}}$$ (gen solution)

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* Note that the "C" from the exponent $e^{\int P dx}$ is not the same as C in solution $y = \dfrac{\int \rho Q dx + C}{\rho}$.
we can pick the "exponent C" to be 0, but we can't do anything with the "solution C"

Application : Mixture

a tank initially contains 40 lbs salt in 400 gals of water.

- Water with $\frac{1}{2}\frac{lb}{gal}$ comes in @ $4\frac{gal}{min}$
- $\boxed{\text{"well-mixed"}}$ solution goes out @ $4\frac{gal}{min}$.

Model amt of salt $A(t)$ in tank over time

$$A' = r_{in}c_{in} - r_{out}\boxed{c_{out}} = r_{in}c_{in} - r_{out}\frac{A(t)}{V(t)}$$

$$\star \quad V(t) = V_0 + (r_{in} - r_{out})t$$

$$r_{in} = 4,\ c_{in} = \frac{1}{2},\ r_{out} = 4,\ A(0) = 40,\ V_0 = 400$$

$$A' = 2 - 4\cdot\frac{A}{400+(4-4)t} = 2 - \frac{4}{400}A$$

$\Rightarrow$ standard form

$$\boxed{A' + \frac{1}{100}A = 2} \qquad P = 1/100,\ Q = 2$$

$$\rho = e^{\int P\,dt} = e^{\int \frac{1}{100}dt} = \boxed{e^{t/100}} \quad \text{(pick constant for } \rho\text{)}$$

$$\rho A = \int \rho Q\,dt + C$$

$$e^{t/100}A = \int 2e^{t/100}\,dt + C$$

$$e^{t/100}A = 200e^{t/100} + C$$

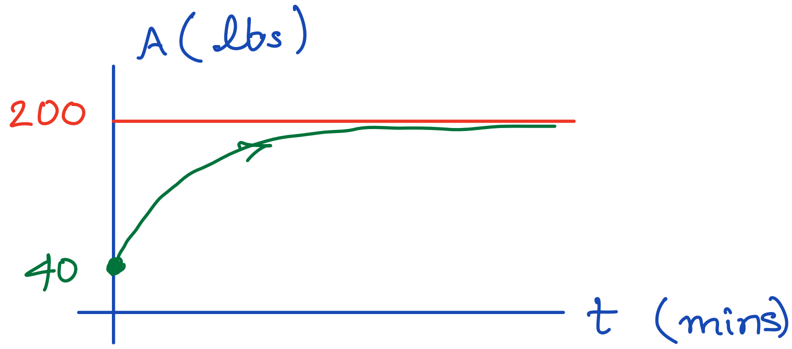
$$\boxed{A = 200 + Ce^{-t/100}}$$ (gen. solution)

need

$$40 = 200 + Ce^{0} \implies C = -160,$$

so particular solution is

$$\boxed{A = 200 - 160e^{-t/100}}$$



• Read example of $r_{out} > r_{in}$ on next page

Example ("continued"): Keeping

$r_{in} = 4, \quad c_{in} = 1/2, \quad A(0) = 40, \quad V(0) = 400,$

now suppose $r_{out} = 6 \frac{gal}{min}$, and again model $A(t)$.

Because $r_{out} > r_{in}$, clearly the tank eventually empties (and so then $A = 0$). What is the max amt of salt in the tank before it empties?

Now $V(t) = 400 + (4-6)t = 400 - 2t$, so clearly the tank empties after 200 mins, so our time range is $\underline{0 \leq t < 200.}$

ODE is $A' = r_{in} c_{in} - r_{out} \frac{A}{V(t)} = 2 - \frac{6}{400-2t} A,$

standard form: $A' + \frac{3}{200-t} A = 2$

$\Longrightarrow \underline{P} = 3/(200-t), \quad Q = 2$

$$\rho = e^{\int P dt} = e^{\int \frac{3}{200-t} dt} = e^{-3\ln(\underline{200-t})} = \boxed{(200-t)^{-3}}$$

note that this is why $0 \leq t < 200$ is relevant b/c we don't want $\ln(-\#)$ or $\ln(0)$

complement ODE and integrate,

$\longrightarrow \rho A = \int \rho Q \, dt + C$

$(200-t)^{-3} A = \int 2(200-t)^{-3} dt + C$

$(200-t)^{-3} A = (200-t)^{-2} + C$

$$\Rightarrow \underline{A = (200-t) + C(200-t)^3} \quad \text{(gen solution)}$$

for particular solution, again use $A(0) = 40$:

$$40 = 200 + C(200)^3$$

$$\Rightarrow \quad C = \frac{-160}{(200)^3} = -\frac{160}{200} \cdot \frac{1}{40{,}000} = \frac{-4}{5} \cdot \frac{1}{40{,}000}$$

$$C = \frac{-1}{50{,}000}.$$

particular solution : $\boxed{A(t) = 200 - t - \frac{1}{50{,}000}(200-t)^3}$

Max in tank? Let $k = 50{,}000$

$$A = 200 - t - \frac{1}{k}(200-t)^3$$

$$A' = -1 - \frac{3}{k}(200-t)^2 \cdot (-1)$$

$$= -1 + \frac{3}{k}(200-t)^2 = 0$$

$$\Rightarrow \frac{3}{k}(200-t)^2 = 1$$

$$(200-t) = \sqrt{k/3}$$

$$\Rightarrow \quad t = 200 - \sqrt{k/3} \approx 70.9 \text{ mins.}$$

at $t = 70.9$, we compute that

$$A \approx 86.07 \text{ lbs}$$